

Non-Linear Dynamics of Continuously Measured Bose-Einstein Condensates in One-Dimensional Harmonic Traps

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Continuous center-of-mass position measurements performed on an interacting harmonically trapped Bose-gas are considered. Using both semi-analytical mean-field approach and completely quantum numerical technique based on positive P-representation, it is demonstrated that the atomic delocalization due to the measurement back action is smaller for a strongly interacting gas. The numerically calculated second-order correlation functions demonstrate appearance of atomic bunching as a result of the center-of-mass measurement. Though being rather small the bunching is present also for strongly interacting gas which is in contrast with the case of unperturbed gas. The performed analysis allows to speculate that for relatively strong interactions the size of atomic cloud determined with a single snapshot measurement can become smaller than the ground-state cloud size.

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I. INTRODUCTION

Now that generation of the trapped degenerate gases has become almost routine procedure in many laboratories the experimental and theoretical activities are directed towards applications of these systems. New frequency standards [1] and various types of weak-force detectors [2, 3] belong to such applications. Another important application of such quantum systems is testing quantum-mechanical predictions in mesoscopic regimes.

All these experiments involve measurements of some property of the trapped gas. As is extensively discussed in quantum theory, the measurements of small objects do inevitably disturb them even if they are prepared as so called quantum non-demolition [4] ones. This issue becomes especially important if the measurement is continuously performed during an experiment. Recently a few experiments have been reported where the conditions required for continuous BEC measurement have been realized [5–8]. One of these works, Ref. [8], especially addresses the measurement back-action on the trapped non-interacting gas. Detailed understanding of measurement back-action is very important for successful realization of quantum-limited feedback control of the trapped gas [9, 10].

An important degree of freedom is the collective or center-of-mass position of the trapped gas. This degree of freedom can be accessed via bringing the system into interaction with a few-mode external field. Hence, this type of measurements is conceptually simpler than a measurement resolving internal structure of the trapped cloud. Moreover, the collective degree of freedom is proven to be experimentally accessible in recent works cited above [5–8]. These facts motivate the theoretical analysis of quantum motion of a degenerate gas with the continuously measured center-of-mass position that we present in this article.

Unlike some other works [2, 8, 10], where similar problems are considered, we focus on a gas of *interacting* particles. It is known that the center-of-mass (CM) motion of a harmonically trapped gas is not coupled to the relative degrees of freedom [11]. This means that the inter-particle interactions do not affect the quantum dynamics of the center of mass. The CM measurement does, however, influence the properties that depend on the relative motion such as particle density distribution [10]. These properties are also affected by the inter-particle interaction. Thus an interesting question arises: does the simultaneous action of the CM position measurement and the interaction result in some nontrivial dynamics of the gas? In other words, how does the interaction strength influence the behavior of the measured gas? It should be mentioned that interesting effects in a different system that also contains an open interacting BEC have recently been discussed in Ref. [12].

Considering a harmonically trapped repulsively interacting 1D Bose gas we show that interaction-induced nonlinearity provides a mechanism that partially compensates measurement back action and stabilizes the cloud spreading. This can be accompanied by the narrowing of the instant density profile of the trapped gas compared with the ground-state density profile. Note that the master equation used in this article to describe the effect of the measurement also describes situations where the CM of the gas is weakly coupled to a hot reservoir [13]. To some extent such coupling might describe quantum fluctuations of the trapping potential that are always present in an experiment. Thus the discussed effects might be relevant even for the cases where there is no explicit observation or control.

We base our discussions on the numerical solution of the many-body quantum problem using positive P-representation of the density operator [14]. Although this approach is known to be limited to relatively small evo-

lution times [15] it seems to be least resource consuming allowing to obtain reasonable results on a single desktop computer. As a complement to numerical results we derive simplified mean-field approach that qualitatively agrees with our numerical results.

The article is organized as follows. In Sec. II the model that is considered is described and examples of possible physical implementations of the CM position measurements are given. The mean-field approach is derived and discussed in Sec. III. Section IV is devoted to the numerical analysis of the system and to the comparison of the results with the mean-field predictions. In Sec. V we present the summary of the results of the article.

II. MODEL

Let us consider a gas of spin-0 bosons with mass m interacting via a repulsive δ -function potential. The gas is trapped in a cylindrically symmetric potential along x axis with frequencies ω_0 and ω_\perp in axial and radial directions, respectively; $a_0 = (m\omega_0)^{-1/2}$ and $a_\perp = (m\omega_\perp)^{-1/2}$ [25] are the corresponding lengths of the harmonic oscillator ground state. In the case of tight confinement in radial direction ($\omega_0 \ll \omega_\perp$) and at low enough temperature all atoms are in the ground state of the radial potential and the gas can be considered as being effectively a trapped 1D gas. In quasi-1D case given that $a_\perp \gg a_0$ the interaction of bosons can be described by the coupling constant $g_{1D} = 2a_s\omega_\perp$ [16], where a_s is the s-wave scattering length.

We assume that the center-of-mass or, more precisely, collective position of the gas in a trap is continuously observed. Strictly speaking one should distinguish between the center-of-mass and collective position [2]. The former is the true mean value that accounts for particle number fluctuations while the latter is merely the weighted sum of particles coordinates. Below we will deal with states having small particle number fluctuations and neglect the mentioned difference.

As a physical model of such a measurement one can suggest, for example, the approach discussed in [17] and experimentally realized in [6, 8]. In these experiment the ^{87}Rb atoms trapped in an atom chip have been coupled to high-finesse optical cavity. The authors show that the atoms can be localized entirely within a single antinode of the standing-wave cavity field. Under these conditions the collective atomic position determines the atom-field coupling strength and can be determined by measuring the cavity transmission.

Another straightforward example is Faraday rotation of the polarization of non-resonant light passing through the gas placed in a non-homogenous magnetic field. Within the framework of the classical theory [19] one can easily find the following expression for the total rotation angle

$$\phi = \frac{2\pi e^3 \omega^2}{m_e^2 c^2 (\omega_R^2 - \omega^2)^2} \int n(x) B(x) dx, \quad (1)$$

where e is the charge of an electron, m_e is the mass of an electron, c is the velocity of light in a vacuum, ω is the probe light frequency, ω_R is the atomic resonance frequency, $n(x)$ is the atom density, $B(x)$ is the magnetic field. Here we integrate over the atomic sample thickness and neglect a deviation of the refractive index from unity. It is seen from Eq. (1) that in the case of the linear magnetic field the Faraday rotation is proportional to the center-of-mass coordinate of the atoms $\bar{X} = (1/N) \int dx x n(x)$, where N is the estimate of the number of atoms that, for simplicity, might be thought of as the average number of atoms. One certainly can think of other schemes to measure the CM coordinate. Therefore, below we do not concentrate on a particular realization but consider the effects typical for all possible experimental approaches to CM position measurements.

Our main concern is the measurement back action, thus we ignore particular measurement outcomes and analyze unconditioned dynamics. In this case the system is described by the master equation

$$\dot{\hat{\rho}} = -i[\hat{H}, \hat{\rho}] - \kappa[\hat{X}, [\hat{X}, \hat{\rho}]]. \quad (2)$$

The first term on the right-hand side of this equation corresponds to the hamiltonian evolution. In second quantization the Hamiltonian has the form

$$\hat{H} = \int \left[-\frac{1}{2m} \hat{\Phi}^\dagger(x) \partial_x^2 \hat{\Phi}(x) + \frac{m\omega_0^2}{2} x^2 \hat{\Phi}^\dagger(x) \hat{\Phi}(x) + g_{1D} \hat{\Phi}^\dagger(x)^2 \hat{\Phi}(x)^2 \right] dx, \quad (3)$$

where $\hat{\Phi}(x)$ and $\hat{\Phi}^\dagger(x)$ are bosonic field operators obeying the commutation relation $[\hat{\Phi}(x), \hat{\Phi}^\dagger(x')] = \delta(x - x')$. The second term in Eq. (2) describes the measurement of the CM coordinate of the atoms

$$\hat{X} = \frac{1}{N} \int x \hat{\Phi}^\dagger(x) \hat{\Phi}(x) dx. \quad (4)$$

The parameter κ characterizes the measurement resolution of the apparatus and determines the back-action strength. Note that the same master equation can be obtained if all the atoms are weakly coupled to the same heat bath of high temperature. Namely, taking these limits into account Eq. (2) follows from the well known Caldera-Legget master equation [13].

III. MEAN-FIELD APPROXIMATION

The evolution of the continuously measured interacting Bose gas can be found from the Hamiltonian (3) and the master equation (2). However, even for the moderate numbers of atoms N the direct numerical integration of the master equation is impracticable due to the large dimensionality of the N -atom Hilbert space. This principle problem of many-particle physics can be attacked with various numerical methods. In this article we use

approach based on positive P-representation of the density operator [14]. However, before presenting numerical results we give a simple mean-field consideration that qualitatively describes the system behavior. It is shown that even this approach reveals some interesting features in the dynamics of the continuously measured interacting Bose gas.

We start by defining a single-atom density matrix as

$$\rho(x_1, x_2) = \text{Tr}\{\hat{\Phi}^\dagger(x_1)\hat{\Phi}(x_2)\hat{\rho}\}. \quad (5)$$

The evolution of the single-atom density matrix is described by the following equation

$$\begin{aligned} \dot{\rho}(x_1, x_2) = & \left\{ i\frac{m}{2}(\partial_{x_2}^2 - \partial_{x_1}^2) - i\frac{m\omega_0^2}{2}(x_2^2 - x_1^2) \right. \\ & + 2ig_{1D}[n(x_1) - n(x_2)] \\ & \left. - \frac{\kappa}{N^2}(x_2 - x_1)^2 \right\} \rho(x_1, x_2), \end{aligned} \quad (6)$$

which was obtained from the master equation (2) neglecting density-density correlations. More precisely, the following approximation has been used

$$\langle \hat{\Phi}^\dagger(x_1)^2 \hat{\Phi}(x_1) \hat{\Phi}(x_2) \rangle \approx (\langle \hat{n}(x_1) \rangle - 1) \rho(x_1, x_2). \quad (7)$$

Note that in the absence of the measurement, that is without the last term in the right-hand side of Eq. (6), this equation is a simple generalization of the well-known Gross-Pitaevskii equation (GPE) for the condensate wave-function. This is easily seen by substituting the coherent state $\rho(x_1, x_2) = \varphi^*(x_1)\varphi(x_2)$ into Eq. (6). The measurement, as can be seen from Eq. (6), results in decay of non-diagonal elements (coherence) of the single-atom density matrix [20]. This measurement-induced decoherence prevents us from describing the gas in terms of the condensate pure-state wave-function.

Using Eq. (6) the evolution of single-atom fluctuations can be derived as

$$\begin{aligned} \partial_t \langle \Delta x^2 \rangle &= \frac{1}{m} \langle \{x, p\} \rangle, \\ \partial_t \langle \Delta p^2 \rangle &= -m\omega_0^2 \langle \{x, p\} \rangle - 4g_{1D} \langle n'(x)p \rangle + \frac{2\kappa}{N^2}, \\ \partial_t \langle \{x, p\} \rangle &= \frac{2}{m} \langle \Delta p^2 \rangle - 2m\omega_0^2 \langle \Delta x^2 \rangle - 4g_{1D} \langle n'(x)x \rangle, \end{aligned} \quad (8)$$

where $\langle \{x, p\} \rangle$ denotes the anticommutator of x and p , $n'(x)$ denotes the derivative of the density distribution with respect to the coordinate. Here we restrict the consideration to the states with $\langle x \rangle = \langle p \rangle = 0$. The system (8) is not closed since it contains the terms proportional to $\langle n'(x)p \rangle$ and $\langle n'(x)x \rangle$ that in general cannot be expressed via the single-atom second moments only.

To render the system (8) closed we perform the following rough approximation, which, as will be seen below, is enough to grasp its qualitative behavior. First we note that without interactions the continuous measurement of the CM coordinate does not change the shape of the distribution function describing the state of the atoms. This

follows directly from the solution of the Fokker-Planck equation (FPE) for the Wigner function $W(x, p)$ in the case of the non-interacting gas

$$\partial_t W(x, p) = \left(-\frac{p}{m} \partial_x + m\omega_0^2 x \partial_p + \frac{\kappa}{N^2} \partial_p^2 \right) W(x, p). \quad (9)$$

Solving this equation analytically with Gaussian initial condition one easily finds that the measurement of BEC only changes the width of the distribution preserving its Gaussian shape. We assume that in the considered situation the distribution also remains approximately Gaussian during the system evolution. This is certainly not true for the strong interaction case [11]. However, for weak interactions this might be a rather good approximation. In this case one obtains the following expressions for the sought averages

$$\begin{aligned} \langle n'(x)p \rangle &\approx -\frac{N}{8\sqrt{\pi} \langle \Delta x^2 \rangle^{3/2}} \langle \{x, p\} \rangle, \\ \langle n'(x)x \rangle &\approx -\frac{N}{4\sqrt{\pi} \langle \Delta x^2 \rangle^{3/2}} \langle \Delta x^2 \rangle. \end{aligned} \quad (10)$$

Substituting this result into the system (8) one obtains the following closed system of equations

$$\begin{aligned} \partial_t \langle \Delta x^2 \rangle &= \frac{1}{m} \langle \{x, p\} \rangle, \\ \partial_t \langle \Delta p^2 \rangle &= -m\Omega_{\text{eff}}^2 \langle \{x, p\} \rangle + \frac{2\kappa}{N^2}, \\ \partial_t \langle \{x, p\} \rangle &= \frac{2}{m} \langle \Delta p^2 \rangle - 4m\Omega_{\text{eff}}^2 \langle \Delta x^2 \rangle, \end{aligned} \quad (11)$$

where the effective frequency Ω_{eff} defined via

$$\Omega_{\text{eff}}^2 = \omega_0^2 - \frac{g_{1D}N}{2\sqrt{\pi}m \langle \Delta x^2 \rangle^{3/2}} \quad (12)$$

has been introduced. The effective frequency is determined by the size of the atomic localization domain, which is represented by $\langle \Delta x^2 \rangle$. The set of nonlinear ordinary equations (11) can easily be solved using one of well established numerical techniques [21]. We use predictor-corrector Adams scheme that is known to be well suited for stiff problems.

To characterize the effect of the CM position measurement on the gas we introduce the so called relative spreading of the atoms η , defined as

$$\eta = \frac{\sqrt{\langle \Delta x^2 \rangle_{\text{meas}}} - \sqrt{\langle \Delta x^2 \rangle_{\text{no-meas}}}}{\sqrt{\langle \Delta x^2 \rangle_{\text{meas}}}}. \quad (13)$$

Here, the subscripts "meas" and "no-meas" are used to distinguish the cases with and without the measurement, respectively. Taking for the initial values of the fluctuations the results obtained from the solution of the time-independent GPE we obtain the results shown in Fig. 1. In this plot the dependence of the relative spreading on the interaction parameter $g_{1D}N$ is shown for two values of measurement strength κ/N^2 . Hereinafter, length is

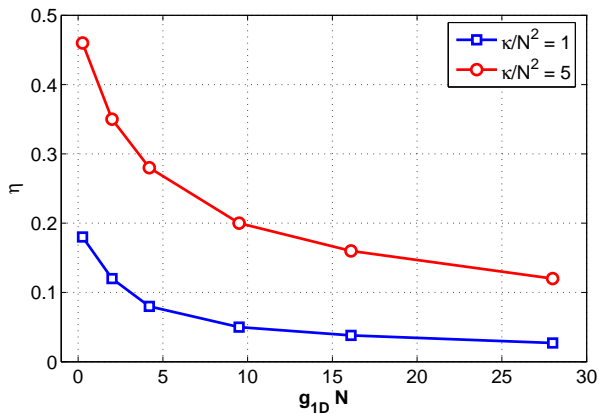


FIG. 1: Relative spreading of the atomic cloud (η) as a function of the interaction strength for $\kappa/N^2 = 1$ (blue squares) and $\kappa/N^2 = 5$ (red circles).

measured in units of a_0 , time in units of $T_0/2\pi$, where T_0 is the period of oscillations in the harmonic potential. Thus, $g_{1D}N$ is the dimensionless quantity. The results of Fig. 1 correspond to the time instant $t = T_0/4$, which is the time when initial momentum uncertainty transforms to the position uncertainty. It is seen that being always positive the relative spreading η goes down for larger values of the interaction constant. This means that the CM measurement always increases the width of the atomic localization domain, but for stronger atomic interactions the measurement-induced spreading is slower. Thus, an interacting trapped gas appears to be more stable against the measurement back-action noise than an ideal one, at least in the beginning of the noise-governed evolution.

A simple explanation of this effect might be obtained on the basis of Eq. (12). The effective frequency represents a degree of atomic localization in an effective potential that is a combination of the trap and mean-field potentials. According to Eq. (12) the measurement induced spreading of the cloud due to non-linear response also increases the effective frequency, which corresponds to better localization. This mechanism partially compensates for atomic delocalization due to the measurement back action. As follows from Eq. (12) the effect should be more pronounced for larger atom-atom coupling. However, the validity of this approach is certainly limited by $g_{1D}N < 2\sqrt{\pi}m\omega_0^2\langle\Delta x^2\rangle^{3/2}$.

There is another interesting feature of the dynamics of an interacting gas subjected to the CM position measurement. Note that the CM of harmonically trapped gas is not coupled to the relative motion [11]. Thus, the measurement-induced delocalization of the CM is increased regardless of atomic interaction strength. The single atom distribution on the contrary depends on the interaction and for strong interaction the measurement-induced change of this quantity can be rather small. This implies appearance of correlations corresponding to

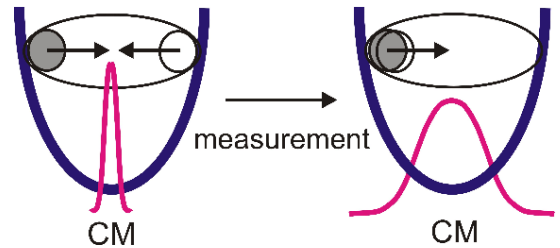


FIG. 2: Two-atom illustration of the measurement-induced narrowing of the instant atomic density profile. The oval symbolizes the region of a single-atom localization that remains almost unchanged. The pulsed-shape distribution represents the CM position uncertainty that gradually increases due to the measurement back action. These two conditions can be simultaneously fulfilled if the atoms tend to bunch together.

bunching of atoms and narrowing of the *instant* density profile compared with the initial unperturbed one.

The easiest way to demonstrate this is to consider only two atoms. Assume that initially each of them spread over the same region in the trap (Fig. 2). For the sake of simplicity let the atoms be spatially anti-correlated (perform anti-phase oscillations). In this case, the size of the atomic cloud is equal to the size of the single-atom localization region, while the CM is strongly localized near the center of the trap. The unconditioned measurement of the CM leads to its delocalization, whereas the size of the atomic cloud, as predicted by the theory described above, remains almost unchanged. As illustrated in Fig. 2 to ensure such dynamics the atoms have to get closer to each other. This might result in an instantaneous density profile that is narrower than the initial ground-state one. In an experiment this effect manifests itself in a size of a single (not ensemble averaged) resonance-image or contrast-image snapshot that should become narrower after some time of the system evolution.

IV. NUMERICAL SIMULATIONS

In this section we discuss the results of *ab initio* numerical simulations of the system. We use the numerical scheme based on the positive P-representation [14]. In spite of known drawbacks [15] of this approach it is efficient for dynamical calculations restricted to relatively short evolution times or small nonlinearities. In addition, this method is relatively easy to implement.

A. Phase-space representation

To treat the problem numerically the continuous atomic distribution is approximated by a lattice model. The space region occupied by the atoms is divided into M equal cells of length Δx : $x_i = i\Delta x$, where $i = 1, \dots, M$.

For each cell i we define annihilation and creation operators

$$\begin{aligned}\hat{a}_i &= \int_{x_i-\Delta x/2}^{x_i+\Delta x/2} dx \hat{\Phi}(x) / \sqrt{\Delta x}, \\ \hat{a}_i^\dagger &= \int_{x_i-\Delta x/2}^{x_i+\Delta x/2} dx \hat{\Phi}^\dagger(x) / \sqrt{\Delta x},\end{aligned}\quad (14)$$

that obey commutation relations $[\hat{a}_i, \hat{a}_j] = 0$ and $[\hat{a}_i, \hat{a}_j^\dagger] = \delta_{ij}$. Then the Hamiltonian (3) can be approximated by the following Bose-Hubbard Hamiltonian

$$\hat{H} = \sum_{ij} \Upsilon_{ij} \hat{a}_i^\dagger \hat{a}_j + \frac{g_{1D}}{\Delta x} \sum_i (\hat{a}_i^\dagger)^2 \hat{a}_i^2. \quad (15)$$

Here the matrix Υ accounts for the kinetic energy and potential energy in the trap, it is defined as

$$\Upsilon_{ij} = \left(\frac{1}{\Delta x^2} + \frac{x_i^2}{2} \right) \delta_{ij} - \frac{1}{2\Delta x^2} \delta_{i+1j} - \frac{1}{2\Delta x^2} \delta_{i-1j}. \quad (16)$$

The discrete version of the non-Hamiltonian term of the master equation (2) is obtained in the similar way and need not be explicitly written.

The total (many-atom) density operator is expanded using a positive P-representation $P^{(+)}$

$$\hat{\rho} = \int P^{(+)}(\mathbf{a}) \hat{\Lambda}(\mathbf{a}) d^{2M} \boldsymbol{\alpha} d^{2M} \boldsymbol{\beta} \quad (17)$$

using the following operator basis

$$\hat{\Lambda}(\mathbf{a}) = \frac{|\boldsymbol{\alpha}\rangle \langle \boldsymbol{\beta}^*|}{\langle \boldsymbol{\beta}^* | \boldsymbol{\alpha} \rangle}. \quad (18)$$

Here, $\mathbf{a} = (\boldsymbol{\alpha}, \boldsymbol{\beta})$, where $\boldsymbol{\alpha} = \{\alpha_1, \dots, \alpha_M\}$ and $\boldsymbol{\beta} = \{\beta_1, \dots, \beta_M\}$ are complex vectors with components $\alpha_i = \alpha'_i + i\alpha''_i$ and $\beta_i = \beta'_i + i\beta''_i$. The $P^{(+)}$ representation is guaranteed to always produce positive-definite diffusion, which is a necessary requirement for a stochastic differential equation.

Substituting the expansion (17) in the master equation (2) and using the standard operator identities [14]

$$\begin{aligned}\hat{\mathbf{a}} \hat{\Lambda}(\mathbf{a}) &= \boldsymbol{\alpha} \hat{\Lambda}(\mathbf{a}), \\ \hat{\mathbf{a}}^\dagger \hat{\Lambda}(\mathbf{a}) &= (\partial_{\boldsymbol{\alpha}} + \boldsymbol{\beta}) \hat{\Lambda}(\mathbf{a}), \\ \hat{\Lambda}(\mathbf{a}) \hat{\mathbf{a}} &= (\partial_{\boldsymbol{\beta}} + \boldsymbol{\alpha}) \hat{\Lambda}(\mathbf{a}), \\ \hat{\Lambda}(\mathbf{a}) \hat{\mathbf{a}}^\dagger &= \boldsymbol{\beta} \hat{\Lambda}(\mathbf{a})\end{aligned}\quad (19)$$

one can obtain the following FPE for the positive $P^{(+)}$ -function

$$\begin{aligned}\frac{\partial}{\partial t} P^{(+)}(\mathbf{a}) &= \left[-\partial_{\boldsymbol{\mu}} A_{\boldsymbol{\mu}} + \frac{1}{2} \partial_{\boldsymbol{\mu}} \partial_{\boldsymbol{\nu}} (\mathbf{B}_{\text{int}} \mathbf{B}_{\text{int}}^T)_{\boldsymbol{\mu}\boldsymbol{\nu}} \right. \\ &\quad \left. + \frac{1}{2} \partial_{\boldsymbol{\mu}} \partial_{\boldsymbol{\nu}} (\mathbf{B}_{\text{meas}} \mathbf{B}_{\text{meas}}^T)_{\boldsymbol{\mu}\boldsymbol{\nu}} \right] P^{(+)}(\mathbf{a}).\end{aligned}\quad (20)$$

Here, $\partial_{\boldsymbol{\mu}}$ denotes partial derivatives $\partial/\partial\alpha_{\boldsymbol{\mu}}$ if $\boldsymbol{\mu} \leq M$ and $\partial/\partial\beta_{\boldsymbol{\mu}-M}$ otherwise, $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$ take values $\boldsymbol{\mu}, \boldsymbol{\nu} =$

$1, \dots, 2M$. The elements of the drift vector $\mathbf{A} = \{A_1, \dots, A_{M+1}, \dots\}$ are given by

$$\begin{aligned}A_i &= -i\Upsilon_{ij}\alpha_j - i\frac{g_{1D}}{\Delta x}\alpha_i^2\beta_i - \frac{\kappa}{N^2}x_i^2\alpha_i, \\ A_{i+M} &= i\Upsilon_{ji}\beta_j + i\frac{g_{1D}}{\Delta x}\beta_i^2\alpha_i - \frac{\kappa}{N^2}x_i^2\beta_i.\end{aligned}\quad (21)$$

The diffusion matrix can be divided into two parts corresponding to different noise sources acting on the atoms. One of these sources may be attributed to the atom-atom interactions. It is described by the diagonal matrix

$$\mathbf{B}_{\text{int}} = \sqrt{\frac{g_{1D}}{\Delta x}} \text{diag}\{(1-i)\alpha_1, \dots, (1+i)\beta_1, \dots\}. \quad (22)$$

The other noise source is the measurement back action. This noise is represented by the matrix \mathbf{B}_{meas} with only one (first) non-zero column. This matrix is written as

$$\mathbf{B}_{\text{meas}} = -\frac{\sqrt{2\kappa}}{N} \begin{pmatrix} x_1\alpha_1 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ x_1\beta_1 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \end{pmatrix}. \quad (23)$$

It is easy to check that the total diffusion matrix is given by $\mathbf{D} = \mathbf{B}_{\text{int}} \mathbf{B}_{\text{int}}^T + \mathbf{B}_{\text{meas}} \mathbf{B}_{\text{meas}}^T$. Assuming that the noise sources discussed above are represented by *independent* Wiener processes one can show that the FPE (20) is equivalent to the set of Itô stochastic differential equations

$$\begin{aligned}d\mathbf{a} &= \mathbf{A}(\mathbf{a}, t)dt + \mathbf{B}_{\text{int}}(\mathbf{a}, t)d\mathbf{W}_{\text{int}}(t) \\ &\quad + \mathbf{B}_{\text{meas}}(\mathbf{a}, t)d\mathbf{W}_{\text{meas}}(t).\end{aligned}\quad (24)$$

For the numerical simulations it is instructive to obtain a set of equations for real functions instead of Eq. (24). To do so we following, for example, Ref. [22] decompose the drift vector and the noise matrices into real and imaginary parts as $\mathbf{A} = \mathbf{A}' + i\mathbf{A}''$ and $\mathbf{B} = \mathbf{B}' + i\mathbf{B}''$. Since $\hat{\Lambda}(\mathbf{a})$ is an analytic function the derivatives $\partial_{\boldsymbol{\alpha}}$ and $\partial_{\boldsymbol{\beta}}$ can be evaluated in either real or imaginary directions so that the resulting drift and diffusion terms can always be made real. Taking this into account one can define the new $4M$ -dimensional real drift vector $\mathbf{A} = \{A'_1, \dots, A'_{M+1}, \dots, A''_1, \dots, A''_{M+1}, \dots\}$ with the elements

$$\begin{aligned}A'_i &= \Upsilon_{ij}\alpha''_j + \frac{g_{1D}}{\Delta x}(n''_i\alpha'_i + n'_i\alpha''_i) - \frac{\kappa}{N^2}x_i^2\alpha'_i, \\ A''_i &= -\Upsilon_{ij}\alpha'_j - \frac{g_{1D}}{\Delta x}(n'_i\alpha'_i - n''_i\alpha''_i) - \frac{\kappa}{N^2}x_i^2\alpha''_i, \\ A'_{i+M} &= -\Upsilon_{ji}\beta''_j - \frac{g_{1D}}{\Delta x}(n''_i\beta'_i + n'_i\beta''_i) - \frac{\kappa}{N^2}x_i^2\beta'_i, \\ A''_{i+M} &= \Upsilon_{ji}\beta'_j + \frac{g_{1D}}{\Delta x}(n'_i\beta'_i - n''_i\beta''_i) - \frac{\kappa}{N^2}x_i^2\beta''_i.\end{aligned}\quad (25)$$

Here, $n'_i = \alpha'_i\beta'_i - \alpha''_i\beta''_i$ and $n''_i = \alpha'_i\beta''_i + \alpha''_i\beta'_i$ are real and imaginary parts of the complex atom number $n_i = n'_i + in''_i$. The new stochastic matrices \mathbf{B}_{int} and \mathbf{B}_{meas} are

$$\mathbf{B}_{\text{int}} = \begin{pmatrix} \emptyset & \mathbf{B}'_{\text{int}} \\ \emptyset & \mathbf{B}''_{\text{int}} \end{pmatrix}, \quad (26)$$

with

$$\begin{aligned}\mathbf{B}'_{\text{int}} &= \sqrt{\frac{g_{1D}}{\Delta x}} \text{diag}\{\alpha'_1 + \alpha''_1, \dots, \beta'_1 - \beta''_1, \dots\}, \\ \mathbf{B}''_{\text{int}} &= \sqrt{\frac{g_{1D}}{\Delta x}} \text{diag}\{-\alpha'_1 + \alpha''_1, \dots, \beta'_1 + \beta''_1, \dots\},\end{aligned}\quad (27)$$

and

$$\mathbf{B}_{\text{meas}} = \begin{pmatrix} \mathbf{O} & \mathbf{B}'_{\text{meas}} \\ \mathbf{O} & \mathbf{B}''_{\text{meas}} \end{pmatrix}, \quad (28)$$

with

$$\mathbf{B}'_{\text{meas}} = \frac{\sqrt{2\kappa}}{N} \begin{pmatrix} -x_1\alpha''_1 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ x_1\beta''_1 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \end{pmatrix} \quad (29)$$

and

$$\mathbf{B}''_{\text{meas}} = \frac{\sqrt{2\kappa}}{N} \begin{pmatrix} x_1\alpha'_1 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -x_1\beta'_1 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \end{pmatrix}. \quad (30)$$

The matrix \mathbf{O} in Eqs. (26) and (28) denotes the $2M \times 2M$ matrix with zero elements.

The SDE (24) is then cast into the following form containing new real $4M$ -dimensional Wiener noise vectors \mathbf{W}_{int} and \mathbf{W}_{meas} :

$$\begin{aligned}d\mathbf{a} &= \mathbf{A}(\mathbf{a}, t)dt + \mathbf{B}_{\text{int}}(\mathbf{a}, t)d\mathbf{W}_{\text{int}}(t) \\ &\quad + \mathbf{B}_{\text{meas}}(\mathbf{a}, t)d\mathbf{W}_{\text{meas}}(t).\end{aligned}\quad (31)$$

The $4M$ dimensional real vector \mathbf{a} is formed of real and imaginary parts of α and β . The elements of noise vectors $d\mathbf{W}_{\text{int}}$ and $d\mathbf{W}_{\text{meas}}$ with the elements denoted by $dW_i^{(\text{int})}$ and $dW_i^{(\text{meas})}$ obey the following properties

$$\begin{aligned}\langle dW_i^{(\text{meas})} \rangle &= \langle dW_i^{(\text{int})} \rangle = 0, \\ \langle dW_i^{(\text{int})} dW_j^{(\text{int})} \rangle &= \delta_{ij}dt, \\ \langle dW_i^{(\text{meas})} dW_j^{(\text{meas})} \rangle &= \delta_{ij}dt, \\ \langle dW_i^{(\text{int})} dW_j^{(\text{meas})} \rangle &= 0, \quad \forall i, j.\end{aligned}\quad (32)$$

Note that the measurement noise matrix \mathbf{B}_{meas} consists of a single non-zero column. This means that all the modes (lattice cells) of the trapped gas are affected by the same measurement-induced noise. This is expectable since the considered noise acts on the collective (CM) degree of freedom of the system. The noise matrix \mathbf{B}_{int} that originates from the atomic interactions is diagonal, which means that each space point of the gas is driven by its individual Wiener noise. Noise sources acting on different coordinates of the gas are statistically independent.

B. Results of numerical simulation

The numerical solution of SDE (31) has been found using semi-implicit method discussed in Ref. [23]. This approach is generally quite stable when applied to non-linear and/or stiff problems. However the problem under consideration seems to have intrinsic instability which manifests itself during the simulations regardless of the stability of used numerical technique. Such an instability, as discussed in Ref. [15], is a result of incorrectly ignored boundary term during the derivation of FPE for the positive P-function.

The conditions when the problem becomes unstable can be grasped looking at the terms of SDE (24) proportional to the interaction constant. As soon as the system evolves to a quantum state with $\alpha^* \neq \beta$ these terms may acquire a real part responsible for exponential growth. Thus considerable (in some sense) deviation of the β from α^* indicates the limits of applicability of the approach.

In a series of numerical experiments performed for different values of the interaction constant the evolution time has been determined during that the dynamics demonstrates no sign of "exploding" trajectories. This "secure" time interval is about quarter of the trap oscillation period. Thus the numerical simulation results presented below correspond to $T_0/4$. The spatial discretization used for the calculation is $\Delta x = 0.33$, the time step is $\Delta t = 10^{-4}$. Further decrease of these values does not change the appearance of the plots presented below. Thus the discretization error can be estimated as being of the order of the line thickness. The number of the stochastic trajectories equals to 20000. The estimated sampling error in this case is about 0.03% which is also below the thickness of the plot lines.

For the initial state of the system it is convenient to take BEC broken-symmetry coherent state. This state for the lattice model reads

$$|\Psi\rangle = |\varphi_{x_1}\rangle \otimes \dots \otimes |\varphi_{x_M}\rangle, \quad (33)$$

where φ_{x_i} is the value of the solution of the time-independent GPE in the space point x_i . The state gives the following initial values for the phase-space variables

$$\begin{aligned}\alpha'_i(0) &= \text{Re}(\varphi_{x_i}), \quad \beta'_i(0) = \alpha'_i(0), \\ \alpha''_i(0) &= \text{Im}(\varphi_{x_i}), \quad \beta''_i(0) = \alpha''_i(0).\end{aligned}\quad (34)$$

The results of numerical calculations of the atom density profile for different values of the interaction strength $g_{1D}N$ are shown in Fig. 3. The curves shown by empty and filled symbols correspond to the situations with and without the measurement, respectively. For small interaction strength the density profiles corresponding to these cases differ substantially (circles). The difference becomes less essential with the increase of $g_{1D}N$ (squares). For yet stronger interaction the effect of the measurement becomes practically negligible, compare curves plotted with filled and empty triangles.

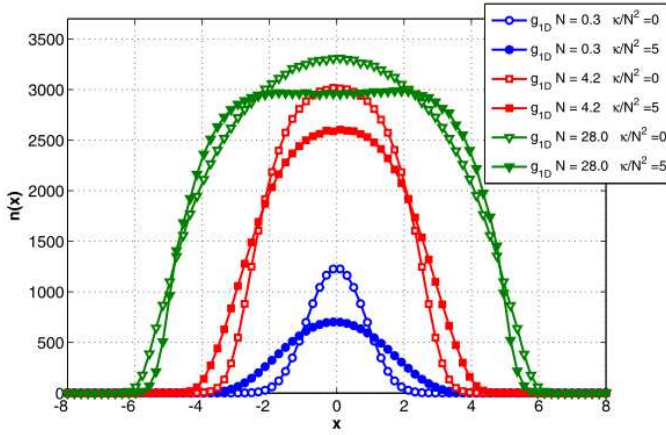


FIG. 3: Atom density profiles for different coupling constants. The curves shown by empty and filled symbols correspond to the situations with and without the measurement, respectively.

Figure 4 compares the relative spreading of the atoms η (13) as a function of the interaction strength for numerical simulation (filled symbols) and the mean-field approach of Sec. III (empty symbols). Curves shown by squares (blue) and circles (red) correspond to different values of κ/N^2 . It is seen that both methods predict similar qualitative behavior of the relative atomic spreading. That is the numerical simulations confirm predicted earlier decrease of spreading η with increasing of atom-atom interaction strength. However, some quantitative discrepancy is observed, which is found to be more pronounced for larger $g_{1D}N$. This is not surprising since in deriving the mean-field approach a couple of not-well-justified assumptions have been made. One of them is the assumption that the density-density type correlations are small. This is exactly the case for the chosen initial state (33) but may become wrong after some time of the system evolution. The other assumption that is certainly violated for a strongly interacting gas is the gaussian density profile approximation. These poor approximations of the mean-field approach result in quantitative difference between the results of the two methods. Nevertheless, the essential feature of interacting gas dynamics subjected to the CM position measurement can be grasped within the mean-field approximation as derived above.

The qualitative arguments presented at the end of Sec. III indicate the possibility of generating bunched states. These states are also characterized by squeezed compared with the unperturbed BEC ground state density profile. Clearly the discussed bunching effect should manifest itself in the second-order correlation function, which can be defined as

$$g_2(x) = \frac{\langle \hat{\Phi}^\dagger(0) \hat{\Phi}^\dagger(x) \hat{\Phi}(x) \hat{\Phi}(0) \rangle}{\langle \hat{\Phi}^\dagger(0) \hat{\Phi}(0) \rangle \langle \hat{\Phi}^\dagger(x) \hat{\Phi}(x) \rangle}. \quad (35)$$

This quantity characterizes density-density correlations

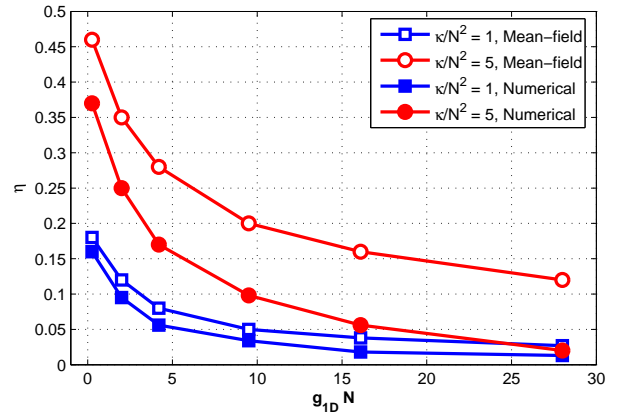


FIG. 4: Relative spreading of the cloud (η) as a function of the interaction strength. Filled and empty symbols show the results of the numerical simulation and the mean-field approximation, respectively. Squares (blue) and circles (red) correspond to different values of the measurement resolution.

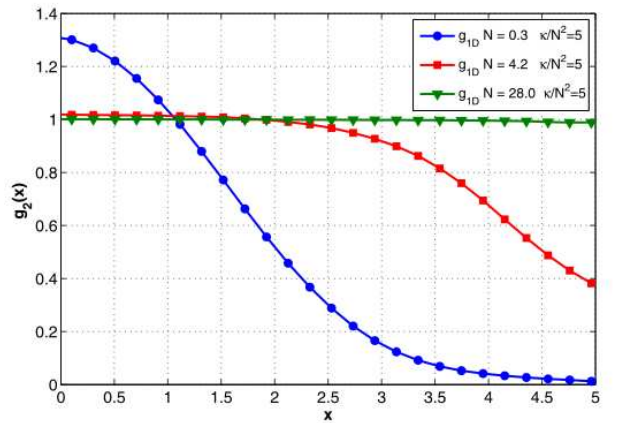


FIG. 5: Second-order correlation functions for $\kappa/N^2 = 5$ and various values of the atom-atom interaction strength $g_{1D}N$.

between the trap center and the point with the coordinate x . In case of independent densities in these points $g_2(x) = 1$, increased (decreased) likelihood to detect atoms separated by x means $g_2(x) > 1$ ($g_2(x) < 1$).

Figure 5 shows these density-density correlations for $\kappa/N^2 = 5$ and different values of atom-atom interactions. It is seen that for small coupling strength $g_{1D}N$ the measurement of the collective coordinate leads to the well observed bunching of the atoms (the curve shown with circles). This indicates that inside the space region occupied by the atoms they are distributed non-uniformly, that is the atoms are grouped in a bunch with the size smaller than that of the occupation region. This is a natural result taking into account that the measurement back-action noise acts on the collective observable result-

ing in the overall delocalization of the atoms while keeping the initial *instant* density distribution.

For larger coupling strength, the curve shown with squares in Fig. 5, the value of the second-order correlation function $g_2(x)$ is smaller for the small separations, $x \rightarrow 0$, than in the previous case. This indicates the smaller degree of bunching that is the result of increased atom-atom repulsion. For yet stronger repulsive interaction (the curve shown with triangles in Fig. 5) the value of the second-order correlation function $g_2(0)$ is only slightly larger than one (cannot be seen on the plot). Thus, the atom density in different locations becomes independent. This is in contrast with the closed strongly interacting gas which exhibits anti-bunching [24].

It is worth noting that for weak (blue curve) and moderate (red curve) interactions the second-order correlation function does not seem to approach unity even for large separations. We believe that this is the result of collective character of considered back action. That means that essentially all the atoms are simultaneously subjected to the same back-action noise, which results in the observed in Fig. 5 long-range correlations.

However we consider a finite system where the limit of very large separations is somewhat ambiguous. At least in numerical calculations of the second-order correlation function we have to restrict the separations to be smaller than the size of the atomic cloud, otherwise the use of definition (35) becomes impractical due to roundoff errors. To summarize the discussion, the presented numerical results only indicate the formation of the long-range correlations. The detailed description of the second-order correlation function in the limit of large separations can be obtained using a properly defined thermodynamic limit.

V. SUMMARY

The CM position measurement of trapped ultra-cold gases can become an important ingredient of various technologies based on ultra-cold atoms. The effect of such a measurement performed on quasi 1D harmonically trapped interacting Bose-gas has been studied in this article. The CM measurement back action disturbs the momentum, which due to the oscillations in the trap results in atomic delocalization. It has been shown that

the interaction-induced nonlinearity can to some extent resist this delocalization decreasing the rate of growth of the width of the average density profile. In other words, the atomic cloud size grows slowly for the stronger interacting gas. This result has first been obtained using a semi-analytical mean-field approach with certain density-density type correlations ignored. Then the same conclusion has been obtained on a more rigorous basis performing numerical simulations based on positive P-representation of the many-atom density operator. In these simulations the atomic correlations have been taken into account, but the continuous density distribution has been approximated by a lattice model.

Numerical simulations show that the averaged atom density profile or atomic cloud size remains almost unchanged for strongly interacting gas during the first quarter of the oscillation period. The numerical approach has allowed to calculate second-order correlation functions in the presence of the CM position measurement. The value of this correlation function for small atomic separation has been found to be greater than one for weakly and even strongly interacting gases, which is the atomic bunching.

In addition, some preliminary conclusions can be made about the width of the instant atomic density profile. As discussed above, the average density profile of strongly interacting gas is almost unchanged in the beginning of the evolution. However, the CM position uncertainty being decoupled from the internal degrees of freedom constantly grows due to measurement back action noise. This is possible if the instant profile shrinks below its initial value. The latter conclusion is to be verified by direct calculations, which will be given elsewhere.

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